10 A conceptual tool for understanding the complexities of mathematical proficiency

Ingrid Rewitzky

Introduction

In Mathematics modules at my institution, we challenge our students not only to know what certain definitions and results hold about a mathematical object but also to know why and to know how to represent and reason mathematically. Our students may convey their personal knowledge of a mathematical object by a picture or an example of their own reasoning process. Through effective teaching and engagement with the rhetoric of Mathematics, these personal representations and intuitive reasoning processes may be shaped into mathematical representations and mathematical reasoning (Ernest 1999). However, many students may resist such 'shaping' with questions such as: 'Where will this mathematical object or result be used in my studies and in real life?' This may seem to be about a particular object, result or topic in a particular branch of Mathematics. For many students, it is the level of abstraction that raises the question and affects mathematical proficiency. When I respond, I rarely provide an actual application but rather illustrate that a particular branch of Mathematics is useful as a whole. For example, calculus contributes to many aspects of daily life, including the design of bridges, the modelling behind weather forecasts or the computer program that determines investment portfolios. However, despite this real-world presence, calculus is also invisible in the sense that the underlying mathematical calculations are not actually seen. Another example is that electronic circuit design found in calculators and mobile phones makes extensive use of so-called imaginary numbers. The informal definition of imaginary numbers is that they are constructed on the basis of something that seems impossible – namely, the square root of minus one. Nevertheless, imaginary numbers have significant uses.

Over the years there have been many different opinions about how to represent all (not only some) mathematical objects. 'Mathematics is the domain within which we find the largest range of semiotic representation systems, both those common to any kind of thinking such as natural language and those specific to Mathematics such as algebraic and formal notations' (Duval 2006: 104). Therefore, Mathematics may be described as

DOI: 10.4324/9781003055549-14

having what Bernstein termed a 'horizontal knowledge structure' with 'strong grammars' – that is, 'a series of specialized languages with specialized modes of interrogation and criteria for the construction and circulation of texts' (2000: 160). The 'horizontal knowledge structure' implies independence of these specialized languages which include probability, algebra, logic, graphs. Each language is unique and does not displace or disprove another language. Moreover, each language has an 'explicit conceptual syntax' (Bernstein 2000: 163) and strong and recognizable principles. This makes it challenging to move between different specialized languages and from informal to formal representations in a specialized language. Mathematical reasoning is how we discover, formulate, justify and generalize claims about mathematical objects. The reasoning techniques are intrinsically linked to the mathematical representation used for referring to the mathematical objects. The complexity of the reasoning may be algorithmic in the case of illustrating that a claim holds for a particular instance of the mathematical object or a formal proof in the case of justifying that a claim holds for the mathematical object.

Throughout its history, Mathematics has been analyzed by philosophers, mathematicians and educationalists in order to better understand the nature of Mathematics and the difficulties experienced by many students in developing a deep and intuitive understanding of Mathematics.

Devlin has argued that we all possess the ability to cope with Mathematics provided we recognize what is required:

> To my mind, a limitation in coping with abstraction presents the greatest barrier to doing mathematics. And yet, as I shall show, the human brain acquired this ability when it acquired language, which everyone has. Thus the reason most people have trouble with mathematics is not that they don't have the ability but that they cannot apply it to mathematical abstractions.
>
> (Devlin 2000: 11)

Abstraction may involve starting with an entity or activity in our reality or world, abstracting the essential idea, features, and structure, understanding these as deeply and completely as possible and then defining a mathematical object and mathematical operations.[1] However, not all Mathematics derives its definition from representations of physicality. Central to Mathematics are generalizations or different levels of abstraction which involve moving from abstract representations to more encompassing abstract representations. Mason (1996) has argued that the ability to generalize is intrinsic to our success in Mathematics because it enhances our capability to apply mathematical concepts across mathematical tasks.

Duval (2006) has described developing mathematical proficiency as the ability (a) to distinguish a mathematical object from its representation, even though the only way that the mathematical object may be accessed is through its representation, and (b) to move between different mathematical representations of the mathematical object. Mathematical proficiency has been defined as having five interdependent components:

> (a) *conceptual understanding* – comprehension of mathematical concepts, operations, and relations (b) *procedural fluency* – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately (c) *strategic competence* – ability to formulate, represent, and solve mathematical problems (d) *adaptive reasoning* – capacity for logical thought, reflection, explanation, and justification, and (e) *productive disposition* – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.
>
> (Kilpatrick *et al.* 2001: 116)

Another description of mathematical proficiency is given in terms of 'what someone knows, can do, and is disposed to do mathematically' (Schoenfeld and Kilpatrick 2008: 326). Furthermore, there is the caution that assessment of mathematical knowledge may be complex, but assessment of strategic competence, adaptive reasoning and the ability to make mathematical connections may be even more difficult (Schoenfeld 2007).

There have been numerous studies on student and societal perceptions of the nature and value of Mathematics and the effect on developing mathematical proficiency. See, for example (MacBean 2004; Wood *et al.* 2012) and other related research described in those papers. MacBean notes

> Many factors affect the quality of student learning. The students' conceptions of and approaches to learning, their prior experiences, perceptions and understanding of their subject, and the teaching and learning context can all influence the learning outcomes achieved.
>
> (MacBean 2004: 553)

MacBean suggests that 'the more students believe that mathematics is integrated and integral to their degree course the more motivated they are likely to be, and the more meaning oriented their approaches to studying will become' (MacBean 2004: 562). Furthermore, Wood suggests that students be encouraged to

> … appreciate the mathematics of all cultures and the contribution of mathematical ideas to the 'business of making accessible the richness of the world we are in, of making dense and substantial our ordinary, day-to-day living in a place – the real work of culture.'
>
> (Wood 2000: 4)

In this chapter, I will illuminate four different insights of mathematical proficiency in terms of *what* mathematical knowledge and *how* one thinks/

reasons mathematically. For the what mathematical knowledge I distinguish between *knowing that* vs *knowing why* and for the *how* one thinks/reasons mathematically I distinguish between thinking/reasoning within Mathematics vs beyond Mathematics. Legitimation Code Theory (LCT) provides conceptual tools to understand these complexities of mathematical proficiency and for a differentiated support model for the sustained improvement of the learning experience in Mathematics.

Methodology

My starting point is the Specialization dimension of LCT, which highlights that every knowledge practice, belief or knowledge claim 'is about or oriented towards something and made by someone' (Maton 2014: 29). The organizing principles of these knowledge-knower structures can be conceptualized as specialization codes, generated by *epistemic relations* and *social relations*. For knowledge claims, epistemic relations are between knowledge and objects towards which the claim is oriented, and social relations are between knowledge and individuals conveying or making the claim. Each of these relations can be stronger or weaker along a continuum. For example, in developing mathematical proficiency, specialized knowledge is emphasized (stronger epistemic relations) while personal experience and opinions of students or lecturers are downplayed (weaker social relations) since not all mathematical ideas can be related to a personal experience or opinion. The two continua together generate *specialization codes*, illustrated in Figure 10.1: *knowledge codes* emphasize specialized knowledge, *knower codes*

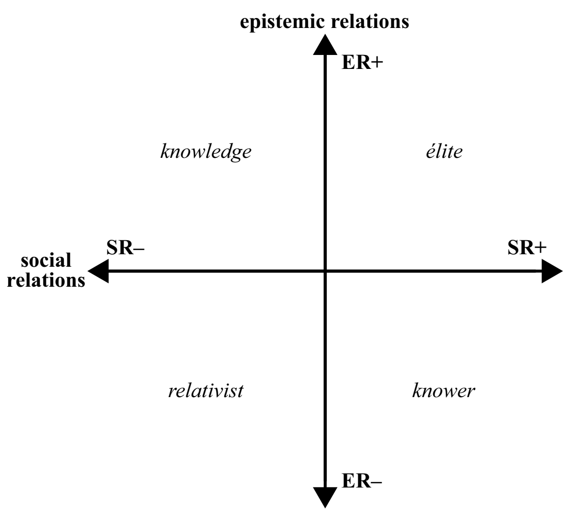

Figure 10.1 The specialization plane (Maton 2014: 30).

emphasize the right kind of knower, *élite codes* emphasize both specialized knowledge and the right kind of knower and *relativist codes* are 'anything goes' (Maton 2014).

The Specialization dimension has provided insights into, for example, degrees of clash between: students' dispositions and educators' pedagogic practices, different approaches within a specific discipline, curriculum and pedagogy of a discipline (Maton 2014, Maton and Chen 2020).

The epistemic relations focus specifically on the nature of knowledge and provide a means to consider the relationship between the *what* and *how* of a knowledge practice. This relationship, explored by the *epistemic plane*, differentiates between *ontic relations* (OR) that represent the strength of relations between a knowledge claim and the object of study and *discursive relations* (DR) that represent the strength of relations between different ways of referring to or dealing with objects of study (Maton 2014). Each relation can be independently stronger (+) or weaker (–) along a continuum. When brought together, the two strengths generate four *insights*. For *purist insights*, practice is based on strong adherence to both a strongly distinguished object of study (OR+) and strongly distinguished approach (DR+). For *doctrinal insights*, practice is not governed by a distinctive object of study (OR–) but by a strongly differentiated approach (DR+). For *situational insights*, knowledge practices are specialized by a distinctive object of study (OR+) and by relative freedom as to how this object is studied (DR–). For *knower/no insights*, practice is either characterized by 'anything goes' (neither a differentiated object nor a differentiated approach; OR–, DR–) or, where these weaker epistemic relations are paired with stronger social relations, legitimated through attributes of the knower.

This *epistemic plane*, together with reflections on my own experience as a mathematician, inspired the theoretical framework I shall use in this chapter. I adapt ontic relations (OR) to refer to '*what* mathematical knowledge' and discursive relations (DR) to refer to '*how* one thinks/reasons.' (This is only one way these concepts can be used and reflect my concerns in this chapter.) It is expected that for a mathematical object of study in a Mathematics module, students' knowledge of a certain mathematical claim about the mathematical object may vary from not knowing that the claim holds to simply knowing that the claim holds to knowing why the claim holds. Also, a student's way of expressing their understanding may have stronger or weaker levels of mathematical formalism. The stronger and weaker ontic relations and discursive relations may be identified along a continuum of mathematical knowledge and a continuum of more or less mathematical formalism, respectively, as in Table 10.1. At right angles to each other, these continua form four quadrants each representing an insight of mathematical proficiency, as previously depicted in Figure 10.2.

Table 10.1 Ontic relations and discursive relations for a particular mathematical object of study

Ontic relations (OR)	Strength of relations between a knowledge claim and the mathematical object of study	+ *stronger*	Know why a mathematical claim holds for the mathematical object of study
		– *weaker*	Know that a mathematical claim holds for the mathematical object of study
		-- weakest	Do not know that a mathematical claim holds for the mathematical object of study
Discursive relations (DR)	Strength of relations between different ways of referring to or reasoning about the mathematical object of study	+ *stronger*	Think/reason with examples, representations, and techniques from mathematics
		– *weaker*	Think/reason with examples, representations, and techniques from beyond mathematics

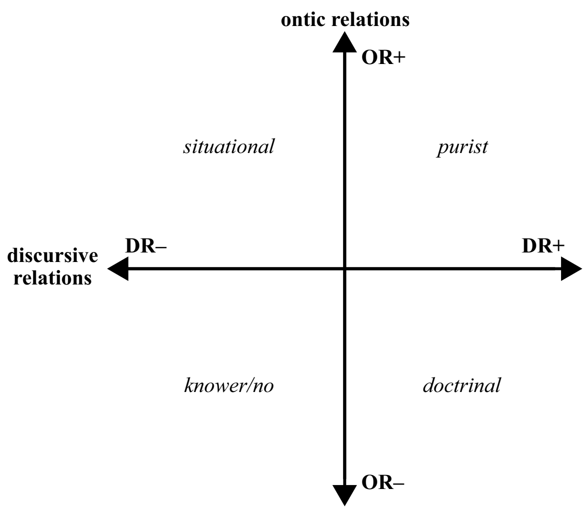

Figure 10.2 The epistemic plane (Maton 2014: 177).

There is no right or wrong *insight*. However, when embarking on a mathematical study, a certain insight may be a preferred starting point over the others. Success in mathematical thinking and reasoning requires 'insight shifting' which involves strengthening or weakening ontic relations and/or discursive relations. In particular,

- *abstract* from personal representation and intuitive reasoning into more mathematical representations and formal reasoning techniques (towards stronger discursive relations);
- *acquire* knowledge of the underlying mathematical ideas and principles (towards stronger ontic relations);
- *generalize* from a specific instance of a mathematical concept or technique to a more encompassing mathematical concept or technique (move towards weaker ontic relations and stronger discursive relations, thereby shifting from situational insight to doctrinal insight);
- *specialize* from a general concept to instances of the concept (move towards weaker ontic relations and so from purist insight to situational insight); and
- *link or apply*, if possible, the mathematical knowledge and skills to a personal experience, perspective or a real-world phenomenon (move to knower/no insight).

Each of these shifts provides a significant challenge when developing mathematical proficiency in a mathematical topic. Different levels of Mathematics proficiency may be described in terms of the different insights navigated. In particular, a basic level of mathematical proficiency will be entirely in *doctrinal insight*; an intermediate level of mathematical proficiency will draw on two different insights – typically, *doctrinal insight* and *situational insight*; and a high level of mathematical proficiency will draw on three different insights as the needs for abstraction, generalization or specialization demand.

Without effective strategies for facilitating insight shifting, there is a potential for clashes. For example, mathematicians may be working in purist insight while most students may be entirely in doctrinal insight or situational insight where less mathematical formalism and rigour are used. Another potential clash arises for mathematical topics that have emerged entirely in purist insight without any link to experiential phenomena.

Analysis of Mathematics proficiency using the epistemic plane

Four case studies have been selected, based on what is studied in a typically undergraduate Mathematics module, namely, mathematical objects, mathematical activities, mathematical representations and mathematical structures. Each will be briefly introduced and an example will be analyzed with the emphasis on the shifting between insights needed for developing mathematical proficiency and the challenges that are typically experienced.

A mathematical object

A mathematical object may be an abstraction of a real-world object or a generalization of an existing mathematical object. As an example, consider the mathematical object called a *function*. At first, a 'function' (a term due to Leibnitz (1646–1716)) simply meant a dependence of numbers given by an analytic expression (situational insight). As the need for formalism and generality grew, the notion of a function went through a gradual transition to the abstract notion of a function between two sets, which was first introduced by Richard Dedekind (1831–1916) (doctrinal insight). However, when setting up a family tree relationship of family members, ancestors and relatives, it will be observed, for example, that the 'mother-of' relationship may relate one family member to more than one other family member (knower/no insight). Such relationships are captured by the abstract notion of binary relation, introduced by Augustus de Morgan (1860) and Peirce's logic of relatives (1870), which generalizes the notion of a function (purist insight). The biggest challenge here is understanding that the purist insight of a function and the doctrinal insight of a function are equivalent. A strategy for developing mathematical proficiency in a mathematical object (in general) is depicted in Figure 10.3 and may involve the following sequence: situational insight to doctrinal insight to knower/no insight to purist insight to doctrinal insight.

A mathematical activity

A mathematical activity typically originates from a practical experience such as, for example, measuring or counting. Let us consider the mathematical activity of finding areas. All first-year Mathematics students will know that the area of a circle may be found by using the formula $A = \pi r^2$ and also know how to use that formula mathematically, for example, to find the area of a circle given its radius or its diameter (doctrinal insight). Unfortunately, few students will know why this is the case. However, people have known, at least since biblical times, that there is a way to divide a cake or a piece of land between two people so that neither is envious of the other – one person cuts and the other chooses (knower/no insight). Properties of the whole may be described in terms of properties of the two parts. Abstracting from this to a circle cut into

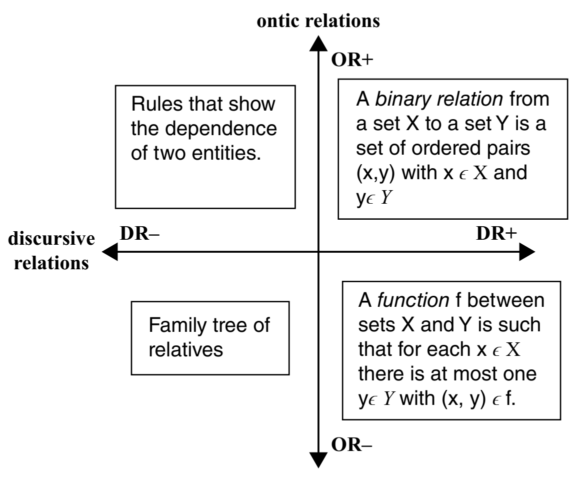

Figure 10.3 Insights for functions.

segments, the area of the circle may be found as the sum of the areas of those segments (situational insight). Reasoning formally in terms of the Riemann sum and the definition of a definite integral yields a formal derivation of the formula (purist insight). Therefore, a strategy for developing mathematical proficiency in a mathematical activity (in general) is depicted in Figure 10.4 and may involve the following sequence: doctrinal insight to knower/no insight to situational insight to purist insight. The biggest challenge is the generalization needed to move from situational insight to purist insight.

Linking two mathematical representations of a mathematical object

A mathematical object may have different mathematical representations, depending on the perspective or branch of Mathematics in which it is being explored. The mathematical object called a torus arises in different branches of Mathematics including calculus and topology and also beyond Mathematics in astrophysics, biology medicine, nuclear physics. Mathematical proficiency in each mathematical representation would be needed before being able to link them. A student may (a) have heard the expression that a coffee cup and doughnut are the same to a mathematician because they both have a single hole (knower/no insight), (b) investigate various torus-shaped objects (situational insight), (c) explore the basic calculus representation of a torus as the surface of revolution generated by revolving a circle in a three-dimensional space (doctrinal insight) and (d) move to a more sophisticated topological representation of a torus as a closed surface with a hole, defined by the product of two circles (purist insight). Therefore, a strategy for developing mathematical proficiency in linking mathematical representations of a mathematical object (in general) is depicted in Figure 10.5 and may involve the following sequence: knower/no insight to situational insight to doctrinal insight to purist insight.

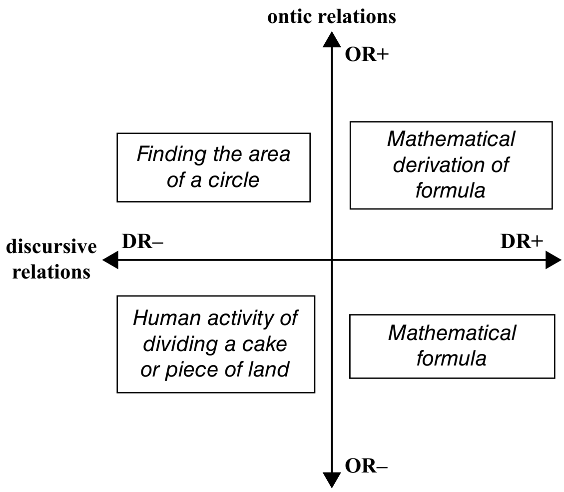

Figure 10.4 Insights for a mathematical activity.

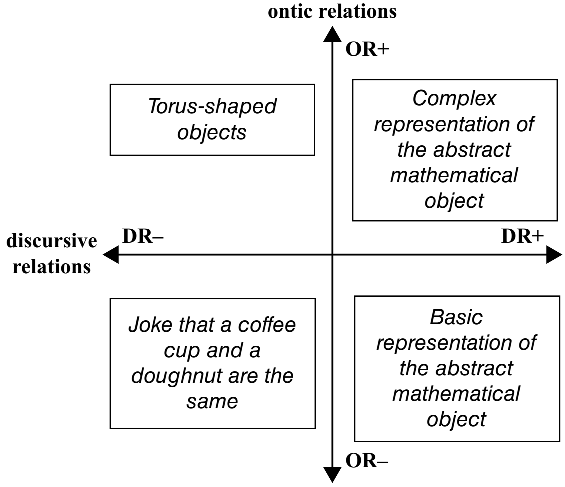

Figure 10.5 Insights for linking mathematical representations of a mathematical object.

The biggest challenges typically arise in generalizing from situational insight to doctrinal insight or in abstracting from doctrinal insight to purist insight.

A mathematical structure

Mathematical structures express mathematical principles or abstractions intended to capture generic properties about a collection of objects (situational insight). A mathematical structure may be likened to a human skeleton. The skeleton is the basic structure of the human body. Although the outward appearances of people may differ, the inward structure, the shape and arrangement of the bones are the same (knower/no insight). Similarly, mathematical structures represent the underlying sameness in situations that may appear outwardly different. Following the Hilbert programme of 1920 and assuming set theory, a mathematical structure is a formal axiomatic system consisting of vocabulary of symbols and connectives, axioms capturing properties of certain symbols and connectives, and rules for combining symbols and connectives and reasoning about them (purist insight). For example, the collection of real numbers with designated symbols 0 and 1, operations of addition and multiplication and axioms of associativity, commutativity, identity is a familiar mathematical structure (doctrinal insight). Therefore, a strategy for developing mathematical proficiency in mathematical structures, depicted in Figure 10.6, may involve linking an informal and a formal yet familiar mathematical structure and then linking the formal yet familiar mathematical theory with formal more abstract mathematical structure.

These strategies may support students to develop mathematical proficiency that has breadth and depth. It will have breadth in the sense of linking different mathematical representations and the associated reasoning techniques

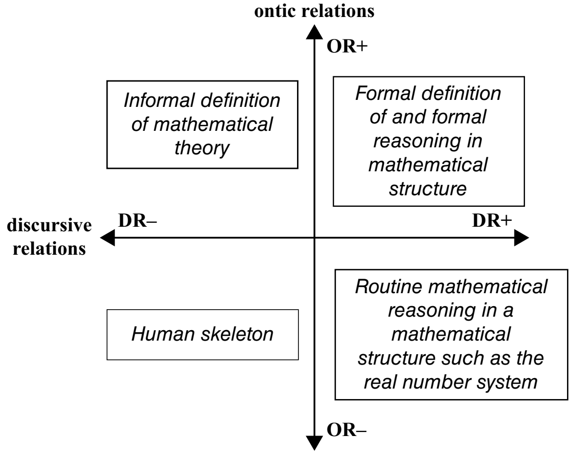

Figure 10.6 Insights for concept of mathematical structure.

at the same level of formalism and of transferring acquired mathematical knowledge and skills within or beyond Mathematics. It will have depth in the sense of understanding and using different levels of mathematical formalism and reasoning techniques.

Discussion and conclusion

Understanding the complexities of Mathematics proficiency of students entering tertiary education has been identified as important for improving student success in STEM programmes (Bohlmann *et al.* 2017; Council on Higher Education 2013; Scott *et al.* 2007). A mathematically proficient student typically has a productive disposition towards Mathematics – that is, the tendency to see sense in Mathematics, to perceive Mathematics as both useful and worthwhile, to believe that steady effort in learning Mathematics pays off and to be an effective doer of Mathematics (Kilpatrick *et al.* 2001).

Using the Specialization dimension of LCT, this chapter illuminates different insights of mathematical proficiency and possible strategies for developing mathematical proficiency necessary for students to be successful in STEM programmes. The key elements studied in Mathematics modules were analyzed in terms of *what* mathematical knowledge and *how* one thinks/reasons, and the emerging insights were depicted on the epistemic plane.

Four key observations may be made. Firstly, moving horizontally, from weaker to stronger discursive relations (understood here as *how* one thinks/reasons), corresponds to the challenge experienced with the nature of abstraction in Mathematics. Secondly, moving vertically, from weaker to stronger ontic relations (understood here as *what* mathematical knowledge), corresponds to the challenge experienced with the level of abstraction in Mathematics. Thirdly, mathematical proficiency presents differently in individual students and corresponds to the ability to shift between different insights. Fourthly, there are potential clashes when the preferred insight of the lecturer and the students differ.

The evidence of challenges with insight shifts and clashes suggests that the Mathematics curriculum could benefit from an analysis of the different insights of the mathematical objects, mathematical activities, mathematical representations and mathematical theories covered and from making these different insights explicit, especially where the biggest challenges are anticipated.

An approach for integrating into the curriculum the insight shifting strategies proposed in this chapter is what may be called a 'differentiated support' model. This model acknowledges different levels of mathematical proficiency of students and offers different levels of support and enrichment so that students have effective learning opportunities to reach their level of success along the path that best suits their style of learning. A key feature of the model is facilitating the navigation between the insights. In particular, for each Mathematics topic, there is a worksheet which is divided into three (or ideally four) parts: procedure problems for checking understanding of core concepts and reasoning for routine problems (doctrinal insight); principle questions for understanding the more theoretical and abstract components of the topic, for developing mathematical writing ability and for thinking critically about results (purist insight); a possibilities section with specific instances or applications to explore, alternative ways to explore a concept and a project (situational insight); and, if possible, reading material of applications in other disciplines or in the real world (knower/no insight). Students are supported and incentivized to submit or present their representation and reasoning approaches to the lecturer or tutor for feedback and there are multiple opportunities for fine-tuning.

This differentiated support model may also provide a valuable framework to guide the planning of technology-mediated support initiatives. Based on the profile of a student, a personalized suite of compulsory and optional learning opportunities for development and growth in mathematical proficiency could be offered throughout the year. It could give students the freedom to choose what they would like to do and when, and develop mathematical proficiency in areas identified through assessments for determining proficiency gaps as well as those that have been self-identified and which are of particular interest to the student.

It is hoped that the insights of this chapter will contribute to improving mathematical proficiency and will be of value to other fundamental disciplines in Science.

Note

1 For example, from the human activity of counting, the mathematical objects called numbers are abstracted together with algebraic operations. Mathematical objects are abstract, unobservable and on the platonic view exist independently. Mathematical representations are the way we refer to mathematical objects. If we think about how to deal with mathematical objects (such as numbers, functions, relations, fields) it is not possible to perceive, manipulate or work with a mathematical object without its mathematical representations. For example, we cannot 'see' a function without its algebraic expression or its graph.

References

Bernstein, B. (2000) *Pedagogy, symbolic control and identity: Theory, research, critique*, Rowman and Littlefield.

Bohlmann, C.A., Prince, R.N. and Deacon, A. (2017) 'Mathematical errors made by high performing candidates writing the national benchmark tests', *Pythagorus – Journal of the Association for Mathematics Education in South Africa,* 38(1): 1–10.

Council on Higher Education. (2013) *A proposal for undergraduate curriculum reform in South Africa: The case for a flexible curriculum structure*, Council on Higher Education.

Devlin, K. (2000) *The math gene: How mathematical thinking evolved and why numbers are like gossip*, Basic Books.

Duval, R. (2006) 'A cognitive analysis of problems of comprehension in a learning of mathematics', *Educational Studies in Mathematics*, 62: 103–131.

Ernest, P. (1999) 'Forms of knowledge in mathematics and mathematics education: Philosophical and rhetorical perspectives', *Educational Studies in Mathematics*, 38: 67–83.

Kilpatrick, J., Swafford, J and Findell, B. (2001) 'The strands of mathematical proficiency', in J. Kilpatrick, J. Swafford, and B. Findell (eds) *Adding it up: Helping children learn mathematics,* National Academy Press.

MacBean, J. (2004) 'Students' conceptions of, and approaches to, studying mathematics as a service subject at undergraduate level', *International Journal of Mathematical Education in Science and Technology*, 35(4): 553–564.

Mason, J. (1996) 'Expressing generality and roots of algebra', in N. Bednarz, C. Kieran, and L. Lee (eds) *Approaches to algebra: Perspectives for research and teaching,* Kluwer Academic Publishers.

Maton, K. (2014) *Knowledge and knowers: Towards a realist sociology of education*, Routledge.

Maton, K. and Chen, R.T-H. (2020) 'Specialization codes: Knowledge, knowers and student success', in J.R. Martin, K. Maton, and Y.J. Doran (eds) *Accessing Academic Discourse: Systemic Functional Linguistics and Legitimation Code Theory*, Routledge.

Scott, I., Yeld, N. and Hendry, J. (2007) A case for improving teaching and learning in South African higher education, Higher Education Monitor No. 6: The Council on Higher Education.

Schoenfeld, A. H. (2007) 'Reflections on an assessment interview: What a close look at student understanding can reveal', in A. H. Schoenfeld (ed) *Assessing Mathematical Proficiency*, Cambridge University Press.

Schoenfeld, A.H. and Kilpatrick, J. (2008) 'Toward a theory of proficiency in teaching mathematics', in T. Wood and D. Tirosh (eds) *International Handbook of Mathematics Teacher Education Volume 2: Tools and Processes in Mathematics Teacher Education*, Sense Publishers.

Wood, L.N. (2000) 'Communicating mathematics across cultures and time', in H. Selin (ed) *Mathematics across Cultures: The History of Non-Western Mathematics*, Kluwer Academic Publishers.

Wood, L.N., Mather, G., Petocz, P., Reid, A., Engelbrecht, J., Harding, A., Houston, K., Smith, G.H. and Perrett, G. (2012) 'University students' views of the role of mathematics in their future', *International Journal of Science and Mathematics Education*, 10(1): 99–119.